

# INCREASING STABILITY FOR THE INVERSE SOURCE SCATTERING PROBLEM WITH MULTI-FREQUENCIES

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**ABSTRACT.** Consider the scattering of the two- or three-dimensional Helmholtz equation where the source of the electric current density is assumed to be compactly supported in a ball. This paper concerns the stability analysis of the inverse source scattering problem which is to reconstruct the source function. Our results show that increasing stability can be obtained for the inverse problem by using only the Dirichlet boundary data with multi-frequencies.

## 1. INTRODUCTION AND PROBLEM STATEMENT

In this paper, we consider the following Helmholtz equation:

$$\Delta u(x) + \kappa^2 u(x) = f(x), \quad x \in \mathbb{R}^d, \quad (1.1)$$

where  $d = 2$  or  $3$ , the wavenumber  $\kappa > 0$  is a constant,  $u$  is the radiated wave field, and  $f$  is the source of the electric current density which is assumed to have a compact support. Denote by  $B_\rho = \{x \in \mathbb{R}^d : |x| < \rho\}$  the ball with radius  $\rho > 0$  and center at the original. Let  $R > 0$  be a constant which is large enough such that  $B_R$  contains the support of  $f$ . Let  $\partial B_R$  be the boundary of  $B_R$ . The following Sommerfeld radiation condition is required to ensure the uniqueness of the wave field  $u$ :

$$\lim_{r \rightarrow \infty} r^{\frac{d-1}{2}} (\partial_r u - i\kappa u) = 0, \quad r = |x|, \quad (1.2)$$

uniformly in all directions  $\hat{x} = x/|x|$ .

For a given function  $u$  on  $\partial B_R$  in two dimensions, it has the Fourier series expansion

$$u(R, \theta) = \sum_{n \in \mathbb{Z}} \hat{u}_n(R) e^{in\theta}, \quad \hat{u}_n(R) = \frac{1}{2\pi} \int_0^{2\pi} u(R, \theta) e^{-in\theta} d\theta.$$

We may introduce the Dirichlet-to-Neumann (DtN) operator  $\mathcal{B} : H^{1/2}(\partial B_R) \rightarrow H^{-1/2}(\partial B_R)$  given by

$$(\mathcal{B}u)(R, \theta) = \kappa \sum_{n \in \mathbb{Z}} \frac{H_n^{(1)'}(\kappa R)}{H_n^{(1)}(\kappa R)} \hat{u}_n(R) e^{in\theta}.$$

For a given function  $u$  on  $\partial B_R$  in three dimensions, it has the Fourier series expansion:

$$u(R, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n \hat{u}_n^m(R) Y_n^m(\theta, \varphi), \quad \hat{u}_n^m(R) = \int_{\partial B_R} u(R, \theta, \varphi) \bar{Y}_n^m(\theta, \varphi) d\gamma.$$

We may similarly introduce the DtN operator  $\mathcal{B} : H^{1/2}(\partial B_R) \rightarrow H^{-1/2}(\partial B_R)$  as follows:

$$(\mathcal{B}u)(R, \theta, \varphi) = \kappa \sum_{n=0}^{\infty} \sum_{m=-n}^n \frac{h_n^{(1)'}(\kappa R)}{h_n^{(1)}(\kappa R)} \hat{u}_n^m(R) Y_n^m(\theta, \varphi).$$

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Here  $H_n^{(1)}$  is the Hankel function of the first kind with order zero,  $h_n^{(1)}$  is the spherical Hankel function of the first kind with order zero,  $Y_n^m$  is the spherical harmonics of order  $n$ , and the bar denotes the complex conjugate. Using the DtN operator, we can reformulate the Sommerfeld radiation condition into a transparent boundary condition

$$\partial_\nu u = \mathcal{B}u \quad \text{on } \partial B_R,$$

where  $\nu$  is the unit outer normal on  $\partial B_R$ . Hence one can also obtain the Neumann data on  $\partial B_R$  once the Dirichlet data is available on  $\partial B_R$ . Now we are in the position to discuss our inverse source problem:

**IP.** Let  $f$  be a complex function with a compact support contained in  $B_R$ . The inverse problem is to determine  $f$  by using the boundary observation data  $u(x, \kappa)|_{\partial B_R}$  with an interval of frequencies  $\kappa \in (0, K)$  where  $K > 1$  is a positive constant.

The inverse source problem has significant applications in medical and biomedical imaging [10], and various tomography problems [1, 14]. In this paper, we study the stability of the above inverse problem. As is known, the inverse source problem does not have a unique solution at a single frequency [7, 9]. Our goal is to establish increasing stability of the inverse problems with multi-frequencies. We refer to [3, 6] for increasing stability analysis of the inverse source scattering problem. In [6], the authors discussed increasing stability of the inverse source problem for the three-dimensional Helmholtz equation in a general domain  $\Omega$  by using the Huygens principle. The observation data are both  $u(x, \kappa)|_{\partial\Omega}$ ,  $0 < \kappa < K$  and  $\nabla u(x, \kappa)|_{\partial\Omega}$ ,  $0 < \kappa < K$ . In [3], the authors studied the stability of the two- and three-dimensional Helmholtz equations via Green's functions. But the stabilities in [3] are different from the stability in this paper where only the Dirichlet data is required. Related results can be found in [12, 13] on increasing stability of determining potentials and in the continuation for the Helmholtz equation. We refer to [4, 8] for a uniqueness result and numerical study for the inverse source scattering problem. A survey can be found in [2] for some general inverse scattering problems with multi-frequencies.

## 2. MAIN RESULT

Let  $0 < r < R$ , define a complex-valued functional space:

$$\mathcal{C}_M = \{f \in H^{n+1}(B_R) : \|f\|_{H^{n+1}(B_R)} \leq M, \text{supp } f \subset B_r \subset B_R, f : B_R \rightarrow \mathbb{C}\},$$

where  $M > 1$  and  $0 < r < R$  are constants. For any  $v \in H^{1/2}(\partial B_R)$ , we set

$$\|v(x, \kappa)\|_{\partial B_R} = \int_{\partial B_R} (|\mathcal{B}v(x, \kappa)|^2 + \kappa^2 |v(x, \kappa)|^2) d\gamma.$$

Now we show the main stability result of the inverse problem.

**Theorem 2.1.** Let  $f_j \in \mathcal{C}_M$ ,  $j = 1, 2$ , and let  $u_j$  be the solution of the scattering problem (1.1)–(1.2) corresponding to  $f_j$ . Then there exists a positive constant  $C$  independent of  $n, K, M, \kappa$  such that

$$\|f_1 - f_2\|_{L^2(B_R)}^2 \leq C \left( \epsilon^2 + \frac{M^2}{\left( \frac{K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{12}}}{(R+1)(6n-6d+3)^3} \right)^{2n-2d+1}} \right), \quad (2.1)$$

where  $K > 1$ ,  $n \geq d$  and

$$\epsilon = \left( \int_0^K \kappa^{d-1} \|(u_1 - u_2)(x, \kappa)\|_{\partial B_R} d\kappa \right)^{\frac{1}{2}}. \quad (2.2)$$

**Remark 2.2.** There are two parts in the stability estimates (2.1): the first part is the data discrepancy and the second part comes from the high frequency tail of the function. It is clear to see that the stability increases as  $K$  increases, i.e., the problem is more stable as more frequencies data are used. We can also see that when  $n < \left[ \frac{K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{12}}}{(R+1)^{\frac{1}{3}}} + d - \frac{1}{2} \right]$ , the stability increases as  $n$  increases, i.e., the problem is more stable as the functions have suitably higher regularity.

Next we prove Theorem 2.1 in the following section.

### 3. PROOF OF THEOREM 2.1

First we present several useful lemmas.

**Lemma 3.1.** *Let  $f_j \in L^2(B_R)$  and  $\text{supp} f_j \subset B_R$ ,  $j = 1, 2$ . Then*

$$\|f_1 - f_2\|_{L^2(B_R)}^2 \leq C \int_0^\infty \kappa^{d-1} \int_{\partial B_R} |\partial_\nu u(x, \kappa) + \kappa u(x, \kappa)|^2 d\gamma d\kappa.$$

*Proof.* Let  $\xi \in \mathbb{R}$  with  $|\xi| = \kappa$ . Multiplying  $e^{-i\xi x}$  on both sides of (1.1) and integrating over  $B_R$ , we obtain

$$\int_{B_R} e^{-i\xi x} f(x) dx = \int_{\partial B_R} e^{-i\xi x} (\partial_\nu u(x, \kappa) + i\xi \nu u(x, \kappa)) d\gamma, \quad |\xi| = \kappa \in (0, \infty).$$

Since  $\text{supp} f \subset B_R$ , we have

$$\int_{\mathbb{R}^d} e^{-i\xi x} f(x) dx = \int_{\partial B_R} e^{-i\xi x} (\partial_\nu u(x, \kappa) + i\xi \nu u(x, \kappa)) d\gamma, \quad |\xi| = \kappa \in (0, \infty),$$

which gives

$$\left| \int_{\mathbb{R}^d} e^{-i\xi x} f(x) dx \right|^2 \leq \left| \int_{\partial B_R} (\partial_\nu u(x, \kappa) + \kappa u(x, \kappa)) d\gamma \right|^2, \quad |\xi| = \kappa \in (0, \infty).$$

Hence,

$$\begin{aligned} & \left( \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{-i\xi x} f(x) dx \right|^2 d\xi \right)^{\frac{1}{2}} \\ & \leq \left( \int_{\mathbb{R}^d} \left| \int_{\partial B_R} (\partial_\nu u(x, \kappa) + \kappa u(x, \kappa)) d\gamma \right|^2 d\xi \right)^{\frac{1}{2}}, \quad |\xi| = \kappa \in (0, \infty). \end{aligned}$$

When  $d = 2$ , we obtain by using the polar coordinates that

$$\begin{aligned} & \left( \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} e^{-i\xi x} f(x) dx \right|^2 d\xi \right)^{\frac{1}{2}} \\ & \leq \left( \int_0^{2\pi} d\theta \int_0^\infty \kappa \left| \int_{\partial B_R} (\partial_\nu u(x, \kappa) + \kappa u(x, \kappa)) d\gamma \right|^2 d\kappa \right)^{\frac{1}{2}} \\ & \leq \left( 2\pi \int_0^\infty \kappa \left| \int_{\partial B_R} (\partial_\nu u(x, \kappa) + \kappa u(x, \kappa)) d\gamma \right|^2 d\kappa \right)^{\frac{1}{2}} \\ & \leq \left( 2\pi^2 R^2 \int_0^\infty \kappa \int_{\partial B_R} |\partial_\nu u(x, \kappa) + \kappa u(x, \kappa)|^2 d\gamma d\kappa \right)^{\frac{1}{2}}, \end{aligned}$$

It follows from the Plancherel theorem that

$$\begin{aligned} \|f_1 - f_2\|_{L^2(B_R)}^2 &= \|f_1 - f_2\|_{L^2(\mathbb{R}^2)}^2 \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\hat{f}_1(\xi) - \hat{f}_2(\xi)|^2 d\xi \\ &\leq C \int_0^\infty \kappa \int_{\partial B_R} |\partial_\nu u(x, \kappa) + \kappa u(x, \kappa)|^2 d\gamma d\xi. \end{aligned}$$

When  $d = 3$ , we obtain by using the polar coordinates that

$$\begin{aligned}
& \left( \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} e^{-i\xi x} f(x) dx \right|^2 d\xi \right)^{\frac{1}{2}} \\
& \leq \left| \int_0^{2\pi} d\theta \int_0^\pi \sin\varphi d\varphi \int_0^\infty \kappa^2 \left| \int_{\partial B_R} (\partial_\nu u(x, \kappa) + \kappa u(x, \kappa)) d\gamma \right|^2 d\kappa \right|^{\frac{1}{2}} \\
& \leq \left( 2\pi^2 \int_0^\infty \kappa^2 \left| \int_{\partial B_R} (\partial_\nu u(x, \kappa) + \kappa u(x, \kappa)) d\gamma \right|^2 d\kappa \right)^{\frac{1}{2}} \\
& \leq \left( \frac{8}{3} \pi^3 R^3 \int_0^\infty \kappa^2 \int_{\partial B_R} |\partial_\nu u(x, \kappa) + \kappa u(x, \kappa)|^2 d\gamma d\kappa \right)^{\frac{1}{2}}.
\end{aligned}$$

It follows from the Plancherel theorem that

$$\begin{aligned}
\|f_1 - f_2\|_{L^2(B_R)}^2 &= \|f_1 - f_2\|_{L^2(\mathbb{R}^3)}^2 \\
&= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |\hat{f}_1(\xi) - \hat{f}_2(\xi)|^2 d\xi \\
&\leq C \int_0^\infty \kappa^2 \int_{\partial B_R} |\partial_\nu u(x, \kappa) + \kappa u(x, \kappa)|^2 d\gamma d\xi,
\end{aligned}$$

which completes the proof.  $\square$

For  $d = 2$ , let

$$\begin{aligned}
I_1(s) &= \int_0^s \kappa^3 \int_{\partial B_R} \left( \int_{B_R} -\frac{i}{4} H_0^{(1)}(\kappa|x-y|)(f_1(y) - f_2(y)) dy \right) \\
&\quad \left( \int_{B_R} \frac{i}{4} \bar{H}_0^{(1)}(\kappa|x-y|)(\bar{f}_1(y) - \bar{f}_2(y)) dy \right) d\gamma(x) d\kappa,
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
I_2(s) &= \int_0^s \kappa \int_{\partial B_R} \left( - \int_{B_R} \frac{i}{4} \partial_\nu H_0^{(1)}(\kappa|x-y|)(f_1(y) - f_2(y)) dy \right) \\
&\quad \left( \int_{B_R} \frac{i}{4} \partial_\nu \bar{H}_0^{(1)}(\kappa|x-y|)(\bar{f}_1(y) - \bar{f}_2(y)) dy \right) d\gamma(x) d\kappa.
\end{aligned} \tag{3.2}$$

For  $d = 3$ , let

$$\begin{aligned}
I_1(s) &= \int_0^s \kappa^4 \int_{\partial B_R} \left( \int_{B_R} \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} (f_1(y) - f_2(y)) dy \right) \\
&\quad \left( \int_{B_R} \frac{e^{-i\kappa|x-y|}}{4\pi|x-y|} (\bar{f}_1(y) - \bar{f}_2(y)) dy \right) d\gamma(x) d\kappa,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
I_2(s) &= \int_0^s \kappa^3 \int_{\partial B_R} \left( \int_{B_R} \partial_\nu \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} (f_1(y) - f_2(y)) dy \right) \\
&\quad \left( \int_{B_R} \partial_\nu \frac{e^{-i\kappa|x-y|}}{4\pi|x-y|} (\bar{f}_1(y) - \bar{f}_2(y)) dy \right) d\gamma(x) d\kappa.
\end{aligned} \tag{3.4}$$

Denote

$$S = \{z = x + iy \in \mathbb{C} : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}.$$

The integrands in (3.1)–(3.4) are analytic functions of  $\kappa$  in  $S$ . The integrals with respect to  $\kappa$  can be taken over any path joining points 0 and  $s$  in  $S$ . Thus  $I_1(s)$  and  $I_2(s)$  are analytic functions of  $s = s_1 + is_2 \in S$ ,  $s_1, s_2 \in \mathbb{R}$ .

**Lemma 3.2.** *Let  $f_j \in L^2(B_R)$ ,  $\text{supp } f_j \subset B_R$ ,  $j = 1, 2$ . We have for any  $s = s_1 + is_2 \in S$  that*

(1) *for  $d = 2$ ,*

$$|I_1(s)| \leq 16\pi^3 R^3 |s|^5 e^{4R|s_2|} \|f_1(x) - f_2(x)\|_{L^2(B_R)}^2, \quad (3.5)$$

$$|I_2(s)| \leq 16\pi^3 R^3 |s|^3 e^{4R|s_2|} \|f_1(x) - f_2(x)\|_{H^1(B_R)}^2, \quad (3.6)$$

(2) *for  $d = 3$ ,*

$$|I_1(s)| \leq 16\pi^3 (|s|^3 R^3 + |s|^4 R^4) e^{4R|s_2|} \|f_1(x) - f_2(x)\|_{L^2(B_R)}^2, \quad (3.7)$$

$$|I_2(s)| \leq 16\pi^3 (|s|^2 R^3 + |s|^3 R^4) e^{4R|s_2|} \|f_1(x) - f_2(x)\|_{H^1(B_R)}^2, \quad (3.8)$$

*Proof.* We first prove (3.7). Let  $\kappa = st$ ,  $t \in (0, 1)$ . A simple calculation yields

$$\begin{aligned} I_1(s) &= \int_0^1 s^5 t^4 \int_{\partial B_R} \left( \int_{B_R} \frac{e^{ist|x-y|}}{4\pi|x-y|} (f_1(y) - f_2(y)) dy \right) \\ &\quad \left( \int_{B_R} \frac{e^{-ist|x-y|}}{4\pi|x-y|} (\bar{f}_1(y) - \bar{f}_2(y)) dy \right) d\gamma(x) dt. \end{aligned}$$

Noting that  $|e^{ist|x-y|}| \leq e^{2R|s_2|}$  for all  $x \in \partial B_R$ ,  $y \in B_R$ , we have

$$\begin{aligned} |I_1(s)| &= \int_0^1 |s|^5 t^4 \int_{\partial B_R} \left| \int_{B_R} \frac{e^{2|s_2|R}}{|x-y|} |f_1(y) - f_2(y)| dy \right|^2 d\gamma(x) dt \\ &\leq \int_0^1 |s|^5 t^4 \int_{\partial B_R} \left| \int_{B_R} |f_1(y) - f_2(y)|^2 dy \right| \int_{B_R} \frac{e^{4R|s_2|}}{|x-y|^2} dy d\gamma(x) dt, \end{aligned}$$

where we have used the Schwarz inequality for the integral with respect to  $y$  in the last inequality. Using the polar coordinates  $\rho = |x - y|$  with respect to  $y$  yields

$$|I_1(s)| \leq \int_0^1 |s|^5 \left( \int_{B_R} |f_1(y) - f_2(y)|^2 dy \right) \int_{\partial B_R} \left( 2\pi^2 \int_0^{2R} e^{4|s_2|R} d\rho \right) d\gamma(x) dt,$$

which implies (3.7).

Next we prove (3.8). Let  $\kappa = st$ ,  $t \in (0, 1)$ . A simple calculation yields

$$\begin{aligned} I_2(s) &= \int_0^1 s^3 t^2 \int_{\partial B_R} \left( \int_{B_R} \partial_\nu \frac{e^{ist|x-y|}}{4\pi|x-y|} (f_1(y) - f_2(y)) dy \right) \\ &\quad \left( \int_{B_R} \partial_\nu \frac{e^{-ist|x-y|}}{4\pi|x-y|} (\bar{f}_1(y) - \bar{f}_2(y)) dy \right) d\gamma(x) dt, \end{aligned}$$

which gives

$$|I_2(s)| = \int_0^1 |s|^3 t^2 \int_{\partial B_R} \left| \int_{B_R} \nabla_x \left( \frac{e^{ist|x-y|}}{|x-y|} \right) \cdot \nu (f_1(y) - f_2(y)) dy \right|^2 d\gamma(x) dt.$$

Noting  $\nabla_x \left( \frac{e^{ist|x-y|}}{|x-y|} \right) = -\nabla_y \left( \frac{e^{ist|x-y|}}{|x-y|} \right)$  and  $\text{supp} f_j \subset B_R, j = 1, 2$ , we have

$$|I_2(s)| = \int_0^1 |s|^3 t^2 \int_{\partial B_R} \left| \int_{B_R} \frac{e^{ist|x-y|}}{|x-y|} \nabla_y (|f_1(y) - f_2(y)| \cdot \nu dy) \right|^2 d\gamma(x) dt.$$

Following a similar argument for proving (3.7), we can prove (3.8).

Now we show the proofs of (3.5) and (3.6). First we prove (3.5). By (3.1) we have

$$I_1(s) = \int_0^1 s^4 t^3 \int_{\partial B_R} \left| \int_{B_R} \frac{i}{4} H_0^{(1)}(st|x-y|) (f_1(y) - f_2(y)) dy \right|^2 d\gamma(x) dt.$$

The Hankel function can also be expressed by the following integral when  $\text{Re} z > 0$  (see e.g., [15], Chapter VI):

$$H_0^{(1)}(z) = \frac{1}{i\pi} \int_{1+\infty i}^1 e^{iz\tau} (\tau^2 - 1)^{-1/2} d\tau.$$

Consequently,

$$\begin{aligned} |H_0^{(1)}(z)| &= \left| \frac{1}{\pi} \int_{+\infty}^0 e^{i(\text{Re} z + i\text{Im} z)(1+ti)} ((1+ti)^2 - 1)^{-1/2} dt \right| \\ &\leq \left| \frac{1}{\pi} e^{i\text{Re} z - \text{Im} z} \int_{+\infty}^0 e^{-t\text{Re} z - it\text{Im} z} (2\tau i - \tau^2)^{-1/2} dt \right| \\ &\leq \frac{1}{\pi} e^{|\text{Im} z|} \int_0^{+\infty} \frac{e^{-t\text{Re} z}}{|\tau^{1/2} (2i - \tau)^{1/2}|} dt \\ &\leq \frac{1}{\pi} e^{|\text{Im} z|} \int_0^{+\infty} \frac{e^{-t\text{Re} z}}{\tau^{1/2} (\tau^2 + 4)^{1/4}} dt \\ &\leq \frac{1}{\pi} e^{|\text{Im} z|} \int_0^{+\infty} \frac{e^{-t\text{Re} z}}{\tau^{1/2} 2^{1/2}} dt \\ &= \frac{1}{\pi} e^{|\text{Im} z|} \left( \int_0^1 \frac{e^{-t\text{Re} z}}{\tau^{1/2} 2^{1/2}} dt + \int_1^{+\infty} \frac{e^{-t\text{Re} z}}{\tau^{1/2} 2^{1/2}} dt \right) \\ &\leq \frac{1}{\pi} e^{|\text{Im} z|} \left( \int_0^1 \frac{1}{\tau^{1/2}} dt + \int_1^{+\infty} e^{-t\text{Re} z} dt \right) \\ &\leq \frac{1}{\pi} e^{|\text{Im} z|} \left( 2 + \frac{1}{\text{Re} z} \right). \end{aligned}$$

Similarly, we can obtain

$$|\overline{H}_0^{(1)}(z)| \leq \frac{1}{\pi} e^{|\text{Im} z|} \left( 2 + \frac{1}{\text{Re} z} \right).$$

Hence we have

$$|I_1(s)| \leq \int_0^1 |s|^4 t^3 \int_{\partial B_R} \left| \int_{B_R} |f_1(y) - f_2(y)|^2 dy \right| \int_{B_R} e^{4R|s_2|} \left( 2 + \frac{1}{|x-y|s_1 t} \right) dy d\gamma(x) dt.$$

Using the polar coordinates  $\rho = |x-y|$  with respect to  $y$  yields

$$|I_1(s)| \leq \int_0^1 |s|^4 t^3 \left| \int_{B_R} |f_1(y) - f_2(y)|^2 dy \right| \int_{\partial B_R} \left( 2\pi^2 \int_0^{2R} e^{4R|s_2|} \left( 2\rho + \frac{1}{s_1 t} \right) d\rho \right) d\gamma(x) dt.$$

which completes the proof of (3.5).

Noting that  $\partial_\nu H_0^{(1)}(\kappa|x-y|) = \nabla_x H_0^{(1)}(\kappa|x-y|) \cdot \nu$  and  $\nabla_x H_0^{(1)}(\kappa|x-y|) = -\nabla_y H_0^{(1)}(\kappa|x-y|)$ , we can prove (3.6) in a similar way.  $\square$

**Lemma 3.3.** *Let  $f_j \in H^n(B_R)$ ,  $n \geq d$ ,  $\text{supp} f_j \subset B_r \subset B_R$ ,  $j = 1, 2$ . Then there exists a constant  $C$  independent of  $n$  such that for any  $s \geq 1$*

$$\int_s^{+\infty} \int_{\partial B_R} \kappa^{d-1} (|\partial_\nu u(x, \kappa)|^2 + \kappa^2 |u(x, \kappa)|^2) d\gamma d\kappa \leq C s^{-(2n-2d+1)} \|f_1 - f_2\|_{H^{n+1}(B_R)}^2. \quad (3.9)$$

*Proof.* It is easy to see that

$$\begin{aligned} & \int_s^{+\infty} \int_{\partial B_R} \kappa^{d-1} (|\partial_\nu u(x, \kappa)|^2 + \kappa^2 |u(x, \kappa)|^2) d\gamma d\kappa \\ &= \int_s^{+\infty} \int_{\partial B_R} \kappa^{d+1} |u(x, \kappa)|^2 d\gamma d\kappa + \int_s^{+\infty} \int_{\partial B_R} \kappa^{d-1} |\partial_\nu u(x, \kappa)|^2 d\gamma d\kappa \\ &\triangleq L_1 + L_2. \end{aligned}$$

Next, we will estimate  $L_1$  and  $L_2$ . When  $d = 3$ , we have

$$\begin{aligned} L_1 &= \int_s^{+\infty} \int_{\partial B_R} \kappa^4 |u(x, \kappa)|^2 d\gamma d\kappa \\ &= \int_s^{+\infty} \int_{\partial B_R} \kappa^4 \left| \int_{\mathbb{R}^3} \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} (f_1 - f_2)(y) dy \right|^2 d\gamma d\kappa. \end{aligned}$$

Using the polar coordinates  $\rho = |y - x|$  originated at  $x$  with respect to  $y$ , we have

$$L_1 = \int_s^{+\infty} \int_{\partial B_R} \kappa^4 \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_0^{+\infty} \frac{e^{i\kappa\rho}}{4\pi} (f_1 - f_2)\rho d\rho \right|^2 d\gamma d\kappa.$$

Using integration by parts and noting  $\text{supp} f_j \subset B_r \subset B_R$ , we obtain

$$L_1 = \int_s^{+\infty} \int_{\partial B_R} \kappa^4 \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_{R-r}^{2R} \frac{e^{i\kappa\rho}}{4\pi(i\kappa)^n} \frac{\partial^n [(f_1 - f_2)\rho]}{\partial \rho^n} d\rho \right|^2 d\gamma d\kappa.$$

Consequently,

$$\begin{aligned}
L_1 &\leq \int_s^{+\infty} \int_{\partial B_R} \kappa^4 \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_{R-r}^{2R} \frac{1}{4\pi\kappa^n} \right. \\
&\quad \left. \left( \left| \sum_{|\alpha|=n} \partial_y^\alpha (f_1 - f_2) \right| \rho + n \left| \sum_{|\alpha|=n-1} \partial_y^\alpha (f_1 - f_2) \right| \right) d\rho \right|^2 d\gamma d\kappa \\
&= \int_s^{+\infty} \int_{\partial B_R} \kappa^4 \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_{R-r}^{2R} \frac{1}{4\pi\kappa^n} \right. \\
&\quad \left. \left( \left| \sum_{|\alpha|=n} \partial_y^\alpha (f_1 - f_2) \right| \frac{1}{\rho} + \left| \sum_{|\alpha|=n-1} \partial_y^\alpha (f_1 - f_2) \right| \frac{n}{\rho^2} \right) \rho^2 d\rho \right|^2 d\gamma d\kappa \\
&\leq \int_s^{+\infty} \int_{\partial B_R} \kappa^4 \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_{R-r}^{2R} \frac{1}{4\pi\kappa^n} \right. \\
&\quad \left. \left( \left| \sum_{|\alpha|=n} \partial_y^\alpha (f_1 - f_2) \right| \frac{1}{R-r} + \left| \sum_{|\alpha|=n-1} \partial_y^\alpha (f_1 - f_2) \right| \frac{n}{(R-r)^2} \right) \rho^2 d\rho \right|^2 d\gamma d\kappa \\
&= \int_s^{+\infty} \int_{\partial B_R} \kappa^4 \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_0^{+\infty} \frac{1}{4\pi\kappa^n} \right. \\
&\quad \left. \left( \left| \sum_{|\alpha|=n} \partial_y^\alpha (f_1 - f_2) \right| \frac{1}{R-r} + \left| \sum_{|\alpha|=n-1} \partial_y^\alpha (f_1 - f_2) \right| \frac{n}{(R-r)^2} \right) \rho^2 d\rho \right|^2 d\gamma d\kappa.
\end{aligned}$$

Changing back to the Cartesian coordinates with respect to  $y$ , we have

$$\begin{aligned}
L_1 &\leq \int_s^{+\infty} \int_{\partial B_R} \kappa^4 \left| \int_{\mathbb{R}^3} \frac{1}{4\pi\kappa^n} \right. \\
&\quad \left. \left( \left| \sum_{|\alpha|=n} \partial_y^\alpha (f_1 - f_2) \right| \frac{1}{R-r} + \left| \sum_{|\alpha|=n-1} \partial_y^\alpha (f_1 - f_2) \right| \frac{n}{(R-r)^2} \right) dy \right|^2 d\gamma d\kappa \\
&\leq Cn \|f_1 - f_2\|_{H^n(B_R)} \int_s^{+\infty} \kappa^{4-2n} d\kappa \\
&= C \frac{n}{2n-5} \|f_1 - f_2\|_{H^n(B_R)} \frac{1}{s^{2n-5}} \\
&\leq 3C \|f_1 - f_2\|_{H^n(B_R)} \frac{1}{s^{2n-5}}, \quad n \geq 3.
\end{aligned} \tag{3.10}$$

Next we estimate  $L_2$  for  $d = 3$ ,

$$\begin{aligned}
L_2 &= \int_s^{+\infty} \int_{\partial B_R} \kappa^2 |\partial_\nu u(x, \kappa)|^2 d\gamma d\kappa \\
&= \int_s^{+\infty} \int_{\partial B_R} \kappa^4 \left| \int_{\mathbb{R}^3} \left( \nabla_y \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} \cdot \nu \right) (f_1 - f_2)(y) dy \right|^2 d\gamma d\kappa.
\end{aligned}$$



Noting that  $\nabla_y \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} = -\nabla_x \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$  and  $\text{supp } f_j \subset B_R$ , we have

$$\begin{aligned} L_2 &= \int_s^{+\infty} \int_{\partial B_R} \kappa^2 |\partial_\nu u(x, \kappa)|^2 d\gamma d\kappa \\ &= \int_s^{+\infty} \int_{\partial B_R} \kappa^2 \left| \int_{\mathbb{R}^3} \left( \nabla_y \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} \cdot \nu \right) (f_1 - f_2)(y) dy \right|^2 d\gamma d\kappa \\ &= \int_s^{+\infty} \int_{\partial B_R} \kappa^2 \left| \int_{\mathbb{R}^3} \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} (\nabla_y (f_1 - f_2)(y) \cdot \nu) dy \right|^2 d\gamma d\kappa. \end{aligned}$$

Following a similar argument as that for the proof of (3.10), we can obtain

$$L_2 \leq Cn \|f_1 - f_2\|_{H^{n+1}(B_R)} \int_s^{+\infty} \kappa^{2-2n} d\kappa = C \frac{n}{2n-3} \|f_1 - f_2\|_{H^{n+1}(B_R)} \frac{1}{s^{2n-3}}, \quad n \geq 2. \quad (3.11)$$

Combining (3.10)–(3.11) and noting  $s > 1$ , we obtain (3.9) for  $d = 3$ .

When  $d = 2$ , we have

$$\begin{aligned} L_1 &= \int_s^{+\infty} \int_{\partial B_R} \kappa^3 |u(x, \kappa)|^2 d\gamma d\kappa \\ &= \int_s^{+\infty} \int_{\partial B_R} \kappa^3 \left| \int_{\mathbb{R}^2} \frac{i}{4} H_0^{(1)}(\kappa|x-y|) (f_1 - f_2)(y) dy \right|^2 d\gamma d\kappa. \end{aligned}$$

The Hankel function can also be expressed by the following integral when  $t > 0$  (e.g., [15], Chapter VI):

$$H_0^{(1)}(t) = \frac{2}{i\pi} \int_0^{+\infty} e^{its} (s^2 - 1)^{-1/2} ds.$$

Using the polar coordinates  $\rho = |y - x|$  originated at  $x$  with respect to  $y$ , we have

$$L_1 = \int_s^{+\infty} \int_{\partial B_R} \kappa^3 \left| \int_0^{2\pi} d\theta \int_0^{+\infty} \frac{1}{4} H_0^{(1)}(\kappa\rho) (f_1 - f_2)\rho d\rho \right|^2 d\gamma d\kappa.$$

Let

$$H_n(t) = \frac{2}{i\pi} \int_0^{+\infty} \frac{e^{its}}{(is)^n (s^2 - 1)^{1/2}} ds, \quad n = 1, 2, \dots. \quad (3.12)$$

It is clear to note that

$$H_0(t) = H_0^{(1)}(t) \quad \text{and} \quad \frac{dH_n(t)}{dt} = H_{n-1}(t), \quad t > 0, \quad n \in \mathbb{N}.$$

Using integration by parts and noting  $\text{supp } f_j \subset B_r \subset B_R$ , we obtain

$$\begin{aligned} L_1 &= \int_s^{+\infty} \int_{\partial B_R} \kappa^3 \left| \int_0^{2\pi} d\theta \int_{R-r}^{2R} \frac{H_1(\kappa\rho)}{4\kappa^2} \frac{\partial(f_1 - f_2)\rho}{\partial\rho} d\rho \right|^2 d\gamma d\kappa \\ &= \int_s^{+\infty} \int_{\partial B_R} \kappa^3 \left| \int_0^{2\pi} d\theta \int_{R-r}^{2R} \frac{H_n(\kappa\rho)}{4\kappa^{n+1}} \frac{\partial^n(f_1 - f_2)\rho}{\partial\rho^n} d\rho \right|^2 d\gamma d\kappa. \end{aligned}$$

Consequently, we have

$$\begin{aligned}
L_1 &\leq \int_s^{+\infty} \int_{\partial B_R} \kappa^3 \left| \int_0^{2\pi} d\theta \int_{R-r}^{2R} \left| \frac{H_n(\kappa\rho)}{4\kappa^{n+1}} \right| \left| \frac{\partial^n(f_1 - f_2)\rho}{\partial \rho^n} \right| d\rho \right|^2 d\gamma d\kappa \\
&\leq \int_s^{+\infty} \int_{\partial B_R} \kappa^3 \left| \int_0^{2\pi} d\theta \int_{R-r}^{2R} \left| \frac{H_n(\kappa\rho)}{4\kappa^{n+1}} \right| \right. \\
&\quad \left. \left( \left| \sum_{|\alpha|=n} \partial_y^\alpha (f_1 - f_2) \right| + \left| \sum_{|\alpha|=n-1} \partial_y^\alpha (f_1 - f_2) \right| \frac{n}{\rho} \right) \rho d\rho \right|^2 d\gamma d\kappa \\
&\leq \int_s^{+\infty} \int_{\partial B_R} \kappa^3 \left| \int_0^{2\pi} d\theta \int_{R-r}^{2R} \left| \frac{H_n(\kappa\rho)}{4\kappa^{n+1}} \right| \right. \\
&\quad \left. \left( \left| \sum_{|\alpha|=n} \partial_y^\alpha (f_1 - f_2) \right| + \left| \sum_{|\alpha|=n-1} \partial_y^\alpha (f_1 - f_2) \right| \frac{n}{R-r} \right) \rho d\rho \right|^2 d\gamma d\kappa.
\end{aligned}$$

Noting (3.12), we see that there exists a constant  $C > 0$  such that  $|H_n(\kappa\rho)| \leq C$  for  $n \geq 1$ . Hence,

$$\begin{aligned}
L_1 &\leq \int_s^{+\infty} \int_{\partial B_R} \kappa^3 \left| \int_0^{2\pi} d\theta \int_{R-r}^{2R} \frac{C}{4\kappa^{n+1}} \right. \\
&\quad \left. \left( \left| \sum_{|\alpha|=n} \partial_y^\alpha (f_1 - f_2) \right| + \left| \sum_{|\alpha|=n-1} \partial_y^\alpha (f_1 - f_2) \right| \frac{n}{R-r} \right) \rho d\rho \right|^2 d\gamma d\kappa.
\end{aligned}$$

Changing back to the Cartesian coordinates with respect to  $y$ , we have

$$\begin{aligned}
L_1 &\leq \int_s^{+\infty} \int_{\partial B_R} \kappa^3 \left| \int_{B_R} \frac{C}{4\kappa^{n+1}} \right. \\
&\quad \left. \left( \left| \sum_{|\alpha|=n} \partial_y^\alpha (f_1 - f_2) \right| + \left| \sum_{|\alpha|=n-1} \partial_y^\alpha (f_1 - f_2) \right| \frac{n}{R-r} \right) dx \right|^2 d\gamma d\kappa \\
&\leq Cn \|f_1 - f_2\|_{H^n(B_R)} \int_s^{+\infty} \kappa^{1-2n} d\kappa = C \frac{n}{2n-2} \|f_1 - f_2\|_{H^n(B_R)} \frac{1}{s^{2n-2}}. \tag{3.13}
\end{aligned}$$

Next we estimate  $L_2$  for  $d = 2$ . A simple calculation yields

$$\begin{aligned}
L_2 &= \int_s^{+\infty} \int_{\partial B_R} \kappa^2 |\partial_\nu u(x, \kappa)|^2 d\gamma d\kappa \\
&= \int_s^{+\infty} \int_{\partial B_R} \kappa^4 \left| \int_{\mathbb{R}^3} \left( \frac{i}{4} \nabla_y H_0^{(1)}(\kappa|x-y|) \cdot \nu \right) (f_1 - f_2)(y) dy \right|^2 d\gamma d\kappa.
\end{aligned}$$

Noting that  $\nabla_y H_0^{(1)}(\kappa|x-y|) = -\nabla_x H_0^{(1)}(\kappa|x-y|)$  and  $\text{supp } f_j \subset B_r \subset B_R$ , we have

$$\begin{aligned}
L_2 &= \int_s^{+\infty} \int_{\partial B_R} \kappa^2 |\partial_\nu u(x, \kappa)|^2 d\gamma d\kappa \\
&= \int_s^{+\infty} \int_{\partial B_R} \kappa^2 \left| \int_{\mathbb{R}^3} \left( \frac{i}{4} \nabla_y H_0^{(1)}(\kappa|x-y|) \cdot \nu \right) (f_1 - f_2)(y) dy \right|^2 d\gamma d\kappa \\
&= \int_s^{+\infty} \int_{\partial B_R} \kappa^2 \left| \int_{\mathbb{R}^3} \frac{i}{4} H_0^{(1)}(\kappa|x-y|) (\nabla_y (f_1 - f_2)(y) \cdot \nu) dy \right|^2 d\gamma d\kappa.
\end{aligned}$$

Following a similar argument as the proof of (3.13), we can obtain

$$\begin{aligned} L_2 &\leq Cn \|f_1 - f_2\|_{H^{n+1}(B_R)} \int_s^{+\infty} \kappa^{-2n} d\kappa \\ &= C \frac{n}{2n-1} \|f_1 - f_2\|_{H^{n+1}(B_R)} \frac{1}{s^{2n-1}}. \end{aligned} \quad (3.14)$$

Combining (3.13) and (3.14) completes the proof of (3.9) for  $d = 2$ . □

The following lemma is proved in [6].

**Lemma 3.4.** *Let  $J(z)$  be analytic in  $S = \{z = x + iy \in \mathbb{C} : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}$  and continuous in  $\bar{S}$  satisfying*

$$\begin{cases} |J(z)| \leq \epsilon, & z \in (0, L], \\ |J(z)| \leq V, & z \in S, \\ |J(0)| = 0. \end{cases}$$

*Then there exists a function  $\mu(z)$  satisfying*

$$\begin{cases} \mu(z) \geq \frac{1}{2}, & z \in (L, 2^{\frac{1}{4}}L), \\ \mu(z) \geq \frac{1}{\pi} \left( \left( \frac{z}{L} \right)^4 - 1 \right)^{-\frac{1}{2}}, & z \in (2^{\frac{1}{4}}L, \infty) \end{cases}$$

*such that*

$$|J(z)| \leq V e^{\mu(z)}, \quad \forall z \in (L, \infty).$$

**Lemma 3.5.** *Let  $f_j \in \mathcal{C}_M, j = 1, 2$ . Then there exists a function  $\mu(z)$  satisfying*

$$\begin{cases} \mu(s) \geq \frac{1}{2}, & s \in (K, 2^{\frac{1}{4}}K), \\ \mu(s) \geq \frac{1}{\pi} \left( \left( \frac{s}{K} \right)^4 - 1 \right)^{-\frac{1}{2}}, & s \in (2^{\frac{1}{4}}K, \infty), \end{cases} \quad (3.15)$$

*such that*

$$|I_1(s) + I_2(s)| \leq CM^2 e^{(4R+1)s} e^{2\mu(s)}, \quad \forall s \in (K, \infty),$$

*for  $d = 2, 3$ .*

*Proof.* It follows from Lemma 3.2 that

$$|[I_1(s) + I_2(s)]e^{-(4R+1)s}| \leq CM^2, \quad \forall s \in S.$$

Recalling (2.2), (3.1)-(3.4), we have

$$|[I_1(s) + I_2(s)]e^{-(4R+1)s}| \leq \epsilon^2, \quad s \in [0, K].$$

A direct application of Lemma 3.5 shows that there exists a function  $\mu(s)$  satisfying (3.15) such that

$$|[I_1(s) + I_2(s)]e^{-(4R+1)s}| \leq CM^2 \epsilon^{2\mu}, \quad \forall s \in (K, \infty),$$

which completes the proof. □

Now we show the proof of Theorem 2.1.

*Proof.* We can assume that  $\epsilon < e^{-1}$ , otherwise the estimate is obvious. Let

$$s = \begin{cases} \frac{1}{((4R+3)\pi)^{\frac{1}{3}}} K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{4}}, & 2^{\frac{1}{4}}((4R+3)\pi)^{\frac{1}{3}} K^{\frac{1}{3}} < |\ln \epsilon|^{\frac{1}{4}}, \\ K, & |\ln \epsilon| \leq 2^{\frac{1}{4}}((4R+3)\pi)^{\frac{1}{3}} K^{\frac{1}{3}}. \end{cases}$$

If  $2^{\frac{1}{4}}(((4R+3)\pi)^{\frac{1}{3}}K^{\frac{1}{3}} < |\ln \epsilon|^{\frac{1}{4}}$ , then we have

$$\begin{aligned} |I_1(s) + I_2(s)| &\leq CM^2 e^{(4R+3)s} e^{-\frac{2|\ln \epsilon|}{\pi}((\frac{s}{K})^4 - 1) - \frac{1}{2}} \\ &\leq CM^2 e^{\frac{(4R+3)}{((4R+3)\pi)^{\frac{1}{3}}} K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{4}} - \frac{2|\ln \epsilon|}{\pi} (\frac{K}{s})^2} \\ &= CM^2 e^{-2\left(\frac{(4R+3)^2}{\pi}\right)^{\frac{1}{3}} K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{2}} \left(1 - \frac{1}{2} |\ln \epsilon|^{-\frac{1}{4}}\right)}. \end{aligned}$$

Noting that  $\frac{1}{2} |\ln \epsilon|^{-\frac{1}{4}} < \frac{1}{2}$ ,  $\left(\frac{(4R+3)^2}{\pi}\right)^{\frac{1}{3}} > 1$  we have

$$|I_1(s) + I_2(s)| \leq CM^2 e^{-K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{2}}}.$$

Using the elementary inequality

$$e^{-x} \leq \frac{(6n-6d+3)!}{x^{3(2n-2d+1)}}, \quad x > 0,$$

we get

$$|I_1(s) + I_2(s)| \leq \frac{CM^2}{\left(\frac{K^2 |\ln \epsilon|^{\frac{3}{2}}}{(6n-6d+3)^3}\right)^{2n-2d+1}}. \quad (3.16)$$

If  $|\ln \epsilon| \leq 2^{\frac{1}{4}}(((4R+3)\pi)^{\frac{1}{3}}K^{\frac{1}{3}}$ , then  $s = K$ . We have from (2.2), (3.1)-(3.4) that

$$|I_1(s) + I_2(s)| \leq \epsilon^2,$$

Here we have noted that for  $s > 0$ ,  $I_1(s) + I_2(s) = \int_0^s \int_{\partial B_R} \kappa^{d-1} (|\partial_\nu u(x, \kappa)|^2 + \kappa^2 |u(x, \kappa)|^2) d\gamma d\kappa$ . Hence we obtain from Lemma 3.3 and (3.16) that

$$\begin{aligned} &\int_0^\infty \int_{\partial B_R} \kappa^{d-1} (|\partial_\nu u(x, \kappa)|^2 + \kappa^2 |u(x, \kappa)|^2) d\gamma d\kappa \\ &\leq I_1(s) + I_2(s) + \int_s^\infty \int_{\partial B_R} \kappa^{d-1} (|\partial_\nu u(x, \kappa)|^2 + \kappa^2 |u(x, \kappa)|^2) d\gamma d\kappa \\ &\leq \epsilon^2 + \frac{CM^2}{\left(\frac{K^2 |\ln \epsilon|^{\frac{3}{2}}}{(6n-6d+3)^3}\right)^{2n-2d+1}} + \frac{C\|f_1 - f_2\|_{H^{n+1}(B_R)}^2}{\left(2^{-\frac{1}{4}}((4R+3)\pi)^{-\frac{1}{3}} K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{4}}\right)^{2n-2d+1}}. \end{aligned}$$

By Lemma 3.1, we have

$$\|f_1 - f_2\|_{L^2(B_R)}^2 \leq C \left( \epsilon^2 + \frac{M^2}{\left(\frac{K^2 |\ln \epsilon|^{\frac{3}{2}}}{(6n-6d+3)^3}\right)^{2n-2d+1}} + \frac{M^2}{\left(\frac{K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{4}}}{(R+1)(6n-6d+3)^3}\right)^{2n-2d+1}} \right).$$

Since  $K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{4}} \leq K^2 |\ln \epsilon|^{\frac{3}{2}}$  when  $K > 1$  and  $|\ln \epsilon| > 1$ , we obtain the stability estimate.  $\square$

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